Synchronization of Networks of Nonlinear Dynamical Systems

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GRK 1362: Cooperative, Adaptive and Responsive Monitoring in Mixed Mode Environments
Challenges in Control of Networked Systems

Complexity Cube [Allgöwer et al., 2009]

**Problem**

Given a group of agents with individual nonlinear dynamics, find control \( u \) such that the state difference \( \|x_i - x_j\| \rightarrow 0 \) for all agents in the group as \( t \rightarrow \infty \).
Outline

1. Introduction to Synchronization

2. Preliminaries
   ▶ Graph Theory
   ▶ System Theory

3. Synchronization of Nonlinear Systems
   ▶ Control Design
   ▶ Examples

4. Conclusion
The Synchronization Problem

- Consider group of three \((i = 1, 2, 3)\) agents each with dynamics
  \[
  \dot{x}_i = u_i, \quad x_i \in \mathbb{R}^2
  \]

- Synchronization means
  \[
  \lim_{t \to \infty} \|x_i - x_j\| = 0, \quad \forall \ i, j \neq i
  \]

- Intuitive nearest neighbor strategy
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- Intuitive **nearest neighbor** strategy
  \[
  u_1 = x_2 - x_1
  \]

  \[
  \Rightarrow x_1 \text{ converges to } x_2
  \]

  \[
  \dot{x}_1 + x_1 = x_2
  \]
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Synchronization means
\[
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\]

Intuitive nearest neighbor strategy
\[
u_2 = x_3 - x_2
\]

\(\Rightarrow \) \(x_2\) converges to \(x_3\)
\[
\dot{x}_2 + x_2 = x_3
\]
The Synchronization Problem

- Consider group of three ($i = 1, 2, 3$) agents each with dynamics
  \[ \dot{x}_i = u_i, \quad x_i \in \mathbb{R}^2 \]

- **Synchronization** means
  \[ \lim_{t \to \infty} \|x_i - x_j\| = 0, \quad \forall \ i, j \neq i \]

- **Intuitive nearest neighbor** strategy
  \[ u_3 = x_2 - x_3 \]

\[ \Rightarrow \quad x_3 \text{ converges to } x_2 \]

\[ \dot{x}_3 + x_3 = x_2 \]
The Synchronization Problem

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Analysis of the Synchronization Problem

- Controls $u_i = x_j - x_i$ depend on information flow between agents $i$ and $j$

$\Rightarrow$ Synchronization implies communication among group members

- Rewrite the result using $x = [x_1^T, x_2^T, x_3^T]^T$

$$\dot{x} = \begin{pmatrix} -x_1 + x_2 \\ -x_2 + x_3 \\ -x_3 + x_2 \end{pmatrix} = \begin{pmatrix} -I_{2\times2} & I_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & -I_{2\times2} & I_{2\times2} \\ 0_{2\times2} & I_{2\times2} & -I_{2\times2} \end{pmatrix} x = -(L \otimes I_{2\times2}) x$$

$\Rightarrow$ Eigenstructure of $L$ determines system behavior [Fax & Murray, 2004]

- Synchronization of nonlinear dynamic systems (unmanned vehicles)
- Provably correct synchronization for such dynamics
- Minimal conditions for synchronization
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Directed Graphs - Digraphs

- Model communication network as digraph $G$
- Vertices = agents and edges = info flow
- Associated Laplacian matrix

\[
L = D_{in} - A^T = \begin{pmatrix}
1 & -1 & 0 \\
0 & 1 & -1 \\
0 & -1 & 1
\end{pmatrix}
\]

- Laplacian is nonnegative matrix
- Smallest eigenvalue $\lambda_0$ always zero
- Associated eigenvector $v_0 = 1^T$
- [Ren & Beard, 2007] $\lambda_0$ simple if $G$
  1. directed spanning tree or
  2. strongly connected

\[
\dot{x} = -(L \otimes I_2)x \\
\Rightarrow \text{span}\{1\} \in \ker(L) \\
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Stability of Nonlinear Systems

Theorem (Lyapunov, 1892)

Given $\dot{x} = f(x)$ with equilibrium $x_e = 0$. Let $V(x)$ be a positive definite function, then $x_e$ is asymptotically stable if $\dot{V}(x) < 0$ for all $x \neq 0$.

+ Stability properties without solving the ODE
  - No systematic method to find $V(x)$

Extension to synchronization [Moreau, 2004]

- Equilibrium $x_e = \Delta x = x_i - x_j$
- Rewrite group dynamics using $\Delta x$
- Show that $\dot{V}(\Delta x) < 0$ for all $\Delta x \neq 0$
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Passivity - An Input-Output Property

- Classical control theory: Input-output descriptions in frequency domain
- Passivity extends this concept to the time domain

Definition (Passivity)
A dynamics $\dot{x} = f(x) + G(x)u$ with output $y = h(x)$ is passive if there exist positive semidefinite functions $V(x)$ and $S(x)$ such that $\dot{V}(x) = u^T y - S(x)$.

- Passive systems never gain energy

\[ u^T y \geq \dot{V}(x) \]

energy put in \hspace{1cm} energy stored

- Find input-output pair and storage function that passifies dynamics

$\Rightarrow$ Stability analysis with $V(x)$ as Lyapunov function
Passivity-Based Coordination

- Passivity useful for coordination [Arcak, 2007], [Chopra & Spong, 2006]
- Consider agents with more complex dynamics
  \[ \dot{x}_i = u_i, \quad y_i = x_i + \dot{x}_i, \quad \text{with} \quad x_i \in \mathbb{R}^2 \]
- Assign preliminary damping \( u_i = -\dot{x}_i + \tau_i \)
- System passive with storage function \( V_i(x_i, \dot{x}_i) = \frac{1}{2} y_i^T y_i \), output \( y_i \), input \( \tau_i \)
- Prove synchronization using
  \[
  V(x, \dot{x}) = \sum_{i=1}^{N} V_i \quad \text{and} \quad \tau_i = -\sum_{j \in \mathcal{N}_i} (y_i - y_j)
  \]
  \[
  \dot{V}(x, \dot{x}) = -S(x, \dot{x}) - y^T (L \otimes I_2) y \leq 0
  \]
  \[
  \geq 0 \quad \geq 0? \]
- Synchronization if \( L = L^T \) and \( L \succeq 0 \) ⇒ Digraph \( \mathcal{G} \) must be balanced
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- Synchronization if \( L = L^\top \) and \( L \succeq 0 \) \( \Rightarrow \) Digraph \( G \) must be balanced
Consider agent dynamics in strict feedback form

\[ \dot{\xi}_{0,i} = a_{0,i}(\xi_{0,i}) + B_{0,i}(\xi_{0,i})\xi_{1,i} \]
\[ \dot{\xi}_{1,i} = a_{1,i}(\xi_{0,i}, \xi_{1,i}) + B_{1,i}(\xi_{0,i}, \xi_{1,i})\xi_{2,i} \]
\[ \vdots \]
\[ \dot{\xi}_{k,i} = a_{k,i}(\xi_{0,i}, \ldots, \xi_{k,i}) + B_{k,i}(\xi_{0,i}, \ldots, \xi_{k,i})u_i \]

**Problem**

Can we find control \( u_i \) to synchronize the group using results from passivity-based coordination?
Synchronization of Nonlinear Systems

System Description

Consider agent dynamics in strict feedback form

\[
\begin{align*}
\dot{\xi}_{0,i} &= a_{0,i}(\xi_{0,i}) + B_{0,i}(\xi_{0,i})\xi_{1,i} \\
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Problem
Can we find control \( u_i \) to synchronize the group using results from passivity-based coordination?
Recursive design

- Use every $\xi_{m,i}, m = 1, \ldots, k$ as virtual control $\alpha_{m,i}$ stabilizing the dynamics
- Introduce error $y_{m,i} = \xi_{m,i} - \alpha_{m,i}$ due to virtuality
- Construct output $y_{k,i}$ rendering the dynamics passive

1. Consider the first subsystem
   \[ \dot{\xi}_{0,i} = a_{0,i} + B_{0,i} \xi_{1,i} \]
   - Choose Lyapunov function $V_{0,i}$
   - Design $\xi_{1,i} = \alpha_{1,i}$ to achieve asymptotic stability
   - Introduce $y_{1,i} = \xi_{1,i} - \alpha_{1,i}$
Synchronization of Nonlinear Systems
Control Design

Recursive design

- Use every $\xi_{m,i}, m = 1, \ldots, k$ as virtual control $\alpha_{m,i}$ stabilizing the dynamics
- Introduce error $y_{m,i} = \xi_{m,i} - \alpha_{m,i}$ due to virtuality
- Construct output $y_{k,i}$ rendering the dynamics passive

2. Extend system to

$$\dot{\xi}_{0,i} = a_{0,i} + B_{0,i}(y_{1,i} + \alpha_{1,i})$$
$$\dot{y}_{1,i} = a_{1,i} - \dot{\alpha}_{1,i} + B_{1,i}\xi_{2,i}$$

- Choose Lyapunov function $V_{1,i} = V_{0,i} + \frac{1}{2}y_{1,i}^T y_{1,i}$
- Design $\xi_{2,i} = \alpha_{2,i}$ to achieve asymptotic stability
- Introduce $y_{2,i} = \xi_{2,i} - \alpha_{2,i}$
Synchronization of Nonlinear Systems
Control Design

Recursive design

- Use every $\xi_{m,i}, m = 1, \ldots, k$ as virtual control $\alpha_{m,i}$ stabilizing the dynamics
- Introduce error $y_{m,i} = \xi_{m,i} - \alpha_{m,i}$ due to virtuality
- Construct output $y_{k,i}$ rendering the dynamics passive

Finally, we have

$$
\dot{y}_{k,i} = a_{k,i} - \dot{\alpha}_{k,i} + B_{k,i}u_i
$$

- Choose Lyapunov function $V_{k,i} = V_{k-1,i} + \frac{1}{2}y_{k,i}^\top y_{k,i}$
- Design $u_i = \beta_i + \tau_i$ to achieve passivity with $V_{k,i}, y_{k,i}, \tau_i$
- Set $\tau_i = -\sum_{j \in N_i} (y_{k,i} - y_{k,j})$ for synchronization
Attitude dynamics of a satellite

\[ \dot{q}_i = \frac{1}{2} \Xi_i(q_i) \omega_i, \]
\[ \dot{\omega}_i = J_i^{-1} (u_i - \omega_i \times J \omega_i) \]

\[ \Rightarrow \omega_i = \alpha_i = -2 \Xi^T_i q_i \]
\[ \Rightarrow y_i = \omega_i - \alpha_i \]

Control design yields

\[ u_i = (\alpha_i + y_i) \times J_i (\alpha_i + y_i) + J_i \left( \frac{1}{2} \Xi_i^T + \dot{\alpha}_i - \frac{1}{4} y_i + \tau_i \right) \]
\[ \tau_i = - \sum_{j \in N_i} (y_i - y_j) \]
Nonholonomic Mobile Robots

- Unicycle dynamics in chained form

\[
\begin{align*}
\dot{\xi}_{0,i} &= v_i \\
\dot{\xi}_{1,i} &= v_i \xi_{2,i} \\
\dot{\xi}_{2,i} &= u_i \\
\Rightarrow v_i &= -\gamma \sum_{j \in N_i} (\xi_{0,i} - \xi_{0,j}) = -\gamma \eta
\end{align*}
\]

- Virtual control \( \xi_{2,i} = \alpha_i = k_1 \frac{\xi_{1,i}}{v_i} \) bounded and converging

- Second control input

\[
u_i = (2\gamma k_1 - 1) v_i \xi_{1,i} - \gamma k_1 v_i^2 y_i + \dot{\alpha}_i - k_2 \sum_{j \in N_i} (y_i - y_j)\]
Summary and Outlook

- Synchronization depends on group communication
- Graph Laplacian determines its properties
- Passivity enables synchronization for complex dynamic systems
- Design for synchronization of a general class of nonlinear systems
- Provably correctness by using Lyapunov analysis

Future Work

- Relax constraints on network topology
- Control design using relative information only
  [Scardovi & Sepulchre, 2008] → linear systems
- Heterogeneous systems
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Bibliography - Fundamentals

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